Quanto Interest-Rate Exchange Options in a Cross-Currency LIBOR Market Model

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Abstract

The purpose of this paper is to price quanto interest-rate exchange options (QIREOs) based on a practical and easy-to-use interest-rate model. A new model, namely the cross-currency LIBOR market model, is used to extend the initial LIBOR market model from a single-currency economy to a cross-currency economy. The cross-currency LIBOR market model is suitable and applicable to pricing a variety of quanto-type interest-rate derivatives. Four different types of quanto interest-rate exchange options are priced in this article. Hedging strategies and calibration procedures are also examined in detail for practical implementation. Furthermore, Monte-Carlo simulation is provided to evaluate the accuracy of the theoretical prices.

Key words: Quanto Interest-Rate Exchange Options, Cross-Currency LIBOR Market Model.

1. Introduction

Quanto interest-rate exchange options (QIREOs), also known as interest-rate difference options, are options written on the difference between two interest rates that are available in different currencies or between two interest rates in one currency, with the final payments made in domestic currency. Interest rate volatility during the past decade has magnified the risk due to an unfavorable shift in the term structure of interest rates, thereby leading to a dramatic increase in the number and types of contingent claims that incorporate options on change in the level of interest rates. These products have been developed to enhance the ability of asset/liability managers to alter their interest-rate exposure. As a result, QIREOs are evolved to exploit interest-rate differentials without directly incurring exchange-rate risk.

The applications of QIREOs are quite extensive and similar to those of differential swaps. However, QIREOs provide more flexibility in certain applications. First, QIREOs provide a mechanism for achieving a payoff based on the differential of interest rates available in two different currencies, which is not directly affected by movements in exchanges rates. Second, as compared with differential swaps, the major advantage of QIREOs is that they can be used to fit a very specific strategy since they can be tailored to provide payoffs that depend on whether the spread of two interest rates is above or below a specified level, or within or outside a specified range on a specific date in the future. Third, QIREOs can provide added precision to a strategy involving differential swaps. For example, a portfolio manager might use a differential swap to capitalize on anticipated yield-curve movements while also purchasing an QIREO on the spread in order to limit his downside risk. Moreover, money market investors may use QIREOs to take advantage of a high-yield currency; asset managers may adopt QIREOs to enhance their portfolio return; liability managers and other borrowers can employ QIREOs to reduce their effective borrowing rates.

Despite the wide applications of QIREOs, the academic literature has paid little attention to how to price such options, especially in the framework of the LIBOR market model. Therefore, the purpose of this article is to price QIREOs based on a practical and easy-to-use interest rate model. In addition, it is worth noting that the well-known “quanto-effect” has to be considered as dealing with foreign assets paid in domestic currency. To achieve this aim, a new model, namely the cross-currency LIBOR market model, is introduced to extend the initial LIBOR market model from a single-currency economy to a cross-currency economy and then adopted to price QIREOs.

This paper employs the cross-currency LIBOR market model to price QIREOs for the following merits. Those interest-rate models that have been developed for pricing interest-rate derivatives can be, loosely speaking, divided into two types: traditional interest-rate models and market models. The traditional interest-rate models, such as the Vasicek model, the Cox-Ingersoll-Ross (CIR) model and the Heath-Jarrow-Morton (HJM) model, describe the behavior of interest rates by specifying market-unobservable and abstract interest rates, such as instantaneous short and forward rates. Contrarily, the LIBOR and swap market models are constructed by specifying market-observable LIBOR and swap rates.

There are some drawbacks to the traditional models. First, because of their abstract and market-unobservable short and forward rates, the underlying market rates, such as LIBOR and swap rates, have to be obtained through a complicated transformation of the abstract rates. Second, the compounding period of their underlying rate is infinitesimal, which contradicts with the market convention of being discretely compounded. Third, caps (floors) and swaptions are the most important and popular interest-rate products that are actively traded in financial markets. The pricing formulae derived from the traditional interest-rate models are incompatible with the widely used Black's formula. As a result, the model calibration procedure is rendered difficulty to execute efficiently. Moreover, most of the traditional models are Gaussian term structure models. As examined in Rogers (1996), Gaussian term structure models have an important theoretical limitation: the rates can attain negative values with positive probability, a tendency which in many cases may cause some pricing errors. In order to improve the aforementioned drawbacks, a new approach to modelling interest-rate behavior has been developed. It is the LIBOR market model (LMM).

The LMM has been developed by Musiela and Rutkowski (1997), Miltersen, Sandmann and Sondermann (1997), and Brace, Gatarek and Musiela (1997, BGM). Because of the following advantages, the model is widely adopted by practitioners. First, the rates modeled are the LIBOR rates, which are market-observable and consistent with the market convention of being discretely compounded. Second, the cap and floor pricing formulae follow the Black's formula, which is consistent with market practice and makes the calibration procedure easier. Moreover, BGM have shown that under the forward measures forward LIBOR rates have a lognormal volatility structure that prevents the forward LIBOR rates from becoming negative with a positive probability. As a result, pricing errors arising from negative rates are avoided.

Furthermore, Wu and Chen (2007) have extended the original BGM model from a single-currency economy to a cross-currency case. They have also incorporated the exchange rate process into the general model setting. Their cross-currency LIBOR market model is very general. Thus, it is suitable to use this model for pricing quanto-type interest-rate derivatives that depend on domestic and foreign interest rates. As a result, the cross-currency LMM will be employed in this article to price four different types of QIREOs.

The remainder of this article is organized as follows. Section 2 briefly describes the development and the framework of the cross-currency LIBOR market model, which is directly drawn from Wu and Chen (2007). Section 3 derives the pricing formulae of the four different types of QIREOs based on the cross-currency BGM (LMM) model. The hedging strategy of each option is also examined. Section 4 provides the calibration procedure for practical implementation and examines the accuracy of the pricing formulae via Monte-Carlo simulation. Section 5 concludes the paper with a brief summary.

2. The Arbitrage-Free Cross-Currency HJM Model and the Arbitrage-Free Cross-Currency BGM Model

In this section, we briefly describe the development and the framework of the cross-currency LIBOR market model as derived by Wu and Chen (2007). Subsection 2.1 establishes an arbitrage-free cross-currency HJM model.[1] Under the arbitrage-free relationship between the drift and the volatility terms in the cross-currency HJM model, an arbitrage-free cross-currency BGM model is also introduced in Subsection 2.2.

2.1 Arbitrage-Free Cross-Currency HJM Model

Assume that trading takes place continuously in time over an interval $\left[0,\tau\right], 0<\tau<\infty$. The uncertainty is described by the filtered probability space $\left(\Omega, F, P, \left\{F\right\}_{t\in[0,\tau]}\right)$ where the filtration is generated by independent standard Brownian motions $\vec{W}(t)=\left(\vec{W}_1(t), \vec{W}_2(t), \ldots, \vec{W}_m(t)\right)$. P represents the actual probability measure. The notations are given below with d for domestic and f for foreign:

$f_k(t,T)$ = the k^{th} country's forward interest rate contracted at time t for instantaneous borrowing and lending at time T with $0 \le t \le T \le \tau$, where $k \in \left\{d, f\right\}$.

$P_k(t,T)$ = the time t price of the k^{th} country's zero coupon bond (ZCB) paying one dollar at time T.

$r_k(t)$ = the k^{th} country's risk-free short rate at time t.

$\beta_k(t)$ = $\exp\left[\int_0^t r_k(u)\,du\right]$, the k^{th} country's money market account at time t with an initial value $\beta_k(0)=1$.

$X(t)$ = the spot exchange rate at $t\in[0,\tau]$ for one unit of foreign currency expressed in terms of domestic currency.

Assumption 2.1 A FAMILY OF FORWARD RATE PROCESS

For any given $T\in[0,\tau]$, the dynamics of the forward rate $f_k(t,T)$, $k\in\{d,f\}$, follows the following process:

$$df_k(t,T)=\mu_{fk}(t,T)\,dt+\sigma_{fk}(t,T)\cdot d\vec{W}(t),\qquad 0\le t\le T\le\tau, \tag{2.1}$$

where $\{f_k(0,T):T\in[0,\tau]\}$ is a nonrandom initial forward curve, $\mu_{fk}(t,T)$ and $\sigma_{fk}(t,T)=(\sigma_{fk1}(t,T),\ldots,\sigma_{fkm}(t,T))$ satisfy some regular conditions.[2]

Equation (2.1) is the notable HJM interest-rate model. Through the various specifications for the volatility coefficients, the random shocks generate significantly different qualitative characteristics of the forward rate processes.

The Zero-Coupon bond price (ZCB) $P_k(t,T)$, $k\in\{d,f\}$, is defined as:

$$P_k(t,T)=\exp\left[-\int_t^T f_k(t,u)\,du\right]. \tag{2.2}$$

From (2.1) and (2.2), the dynamics of the ZCB price can be derived as given below

$$\frac{dP_k(t,T)}{P_k(t,T)}=\left[r_k(t)+b_k(t,T)\right]dt-\sigma_{Pk}(t,T)\cdot d\vec{W}(t),\qquad 0\le t\le T\le\tau, \tag{2.3}$$

where $\sigma_{Pk}(t,T)=(\sigma_{Pk1}(t,T),\ldots,\sigma_{Pkm}(t,T))$ with

$$\sigma_{Pki}(t,T)=\int_t^T \sigma_{fki}(t,u)^2\,du\qquad \text{for}\quad i=1,2,\ldots,m$$

and
$$b_k(t,T)=-\int_t^T \mu_{fk}(t,u)\,du+\frac{1}{2}\left\|\sigma_{Pk}(t,T)\right\|^2.$$

Assumption 2.2 THE SPOT EXCHANGE RATE DYNAMICS

The dynamics of the spot exchange rate $X(t)$ is given as follows:

$$dX(t)=X(t)\mu_X(t)\,dt+X(t)\sigma_X(t)\,d\vec{W}(t), \tag{2.4}$$

where $\mu_X(t)$ and $\sigma_X(t,T)=(\sigma_{X1}(t),\ldots,\sigma_{Xm}(t))$ satisfy some regular conditions.[3]

For greater flexibility, the number of the random shocks, m, are not designated exactly. Rather, they are stipulated dependent on the simplicity and accuracy required by the user. For example, six random shocks may be used to capture all factors causing the stochastic behaviors of the entire forward rate curve and the spot exchange rate. The first two random shocks can be interpreted, respectively, as the short-term and the long-term factors causing the shift of different maturity ranges on the domestic term structure. Similarly, the third and fourth shocks have the same effects on the foreign case. The correlation between the domestic and the foreign term structure is affected by the fifth shock. The remaining shock can be interpreted as the factor that causes unanticipated movements in the exchange rate.

In order to make the economy both complete and arbitrage (i.e., there exists a unique martingale measure[4]), some conditions are imposed upon the previous dynamics. Under these conditions, the volatility terms of all the stochastic processes remain unchanged, but the drift terms become some special structures. These corresponding relationships between the drift and the volatility terms will be employed in the next section to derive the cross-currency LIBOR market model.

To determine the unique domestic martingale measure,[5] all assets must be denominated in domestic currency. Therefore, the foreign assets must be denominated in domestic currency and regarded as the 'general' domestic assets. Define $P_f^*(t,T) = P_f(t,T)X(t)$, $\beta_f^*(t,T) = \beta_f(t)X(t)$. Then, all the domestic-currency-denominated assets are discounted by the domestic money market account and listed as follows:

$$Z_{Pd}(t,T) = \frac{P_d(t,T)}{\beta_d(t)}, \qquad Z_{Pf}(t,T) = \frac{P_f^*(t,T)}{\beta_d(t)}, \qquad Z_{rf}(t,T) = \frac{\beta_f^*(t,T)}{\beta_d(t)}.$$

By Ito's lemma, the stochastic processes of these domestic-currency-denominated assets can be expressed as follows:

$$\frac{dZ_{Pd}(t,T)}{Z_{Pd}(t,T)} = b_d(t,T)dt - \sigma_{Pd}(t,T)\cdot d\vec{W}(t)$$
$$\frac{dZ_{Pf}(t,T)}{Z_{Pf}(t,T)} = b_f^*(t,T)dt + \left[\sigma_X(t) - \sigma_{Pf}(t,T)\right]\cdot d\vec{W}(t)$$
$$\frac{dZ_{rf}(t,T)}{Z_{rf}(t,T)} = \mu_{\beta f}^*(t,T)dt + \sigma_X(t)\cdot d\vec{W}(t)$$

where

$$b_f^*(t,T) = r_f(t) + b_f(t,T) + \mu_X(t) - \sigma_X(t)\cdot\sigma_{Pf}(t,T) - r_d(t)$$
$$\mu_{\beta f}^*(t) = r_f(t) + \mu_X(t) - r_d(t)$$

Since the model has m random shocks, m distinct assets are needed to hedge against these risks. The (m-2) domestic ZCBs with different maturities, the domestic-currency-dominated foreign ZCB, and the foreign money market account are chosen.

By citing Wu and Chen (2007), the dynamics of the forward rates and the exchange rate under the domestic martingale measure Q are presented by Proposition 2.1 as given below.

Proposition 2.1 THE DYNAMICS UNDER THE DOMESTIC MARTINGALE MEASURE

Under the domestic martingale measure Q, for any $T \in [0, \tau]$, *the dynamics of the forward rates and the exchange rate are given as follows:*

$$\begin{aligned}
df_d(t,T) &= \sigma_{fd}(t,T) \cdot \sigma_{Pd}(t,T)dt + \sigma_{fd}(t,T) \cdot dW(t) \\
df_f(t,T) &= \sigma_{ff}(t,T) \cdot \left[\sigma_{Pf}(t,T) - \sigma_X(t)\right]dt + \sigma_{ff}(t,T) \cdot dW(t) \\
\frac{dX(t)}{X(t)} &= \left[r_d(t) - r_f(t)\right]dt + \sigma_X(t) \cdot dW(t)
\end{aligned}$$

where the first subscript of σ_{fd} *and* σ_{ff} *denotes the forward rate while the second represents the country, either domestic or foreign.*

It is worth emphasizing that even if the more general HJM model is considered, *the drift restriction of the domestic forward rate for no-arbitrage* still remain unchanged. However, for the foreign case, the drift appears to have one additional term, $\sigma_{ff}(t,T) \cdot \sigma_X(t)$, which specifies the instantaneous correlation between the exchange rate and the foreign forward rate. It is also observed that the drift terms of the foreign assets are augmented by the instantaneous correlations between the exchange rate and the assets.

These arbitrage-free relationships between the volatility and the drift terms as given in Proposition 2.1 can be employed to derive the arbitrage-free cross-currency BGM model and then applied to pricing cross-currency derivatives.

2.2 Arbitrage-Free Cross-Currency BGM Model

It is important to note that, thereafter, the term structure of interest rates is modeled by specifying the LIBOR rates dynamics, rather than the forward rates dynamics. However, we still use the same notations, the same economic environment, and the arbitrage-free relationships between the drift and the volatility terms as given in Proposition 2.1 to derive the cross-currency BGM model under the martingale measure.

For some $\delta > 0$, $T \in [0,\tau]$ and $k \in \{d, f\}$, define the forward LIBOR rate process $\{L_k(t,T); 0 \le t \le T\}$ as given by

$$1+\delta L_k(t,T) = \frac{P_k(t,T)}{P_k(t,T+\delta)} = \exp\left(\int_T^{T+\delta} f_k(t,u)\,du\right) \tag{2.5}$$

Wu and Chen (2007) have shown that the dynamics of the forward LIBOR rates and the exchange rate under the domestic spot martingale measure Q can be expressed by Proposition 2.2 as follows.

Proposition 2.2 THE CROSS-CURRENCY LIBOR MARKET MODEL UNDER THE DOMESTIC SPOT MARTINGALE MEASURE

Under the domestic spot martingale measure Q, the processes of the forward LIBOR rates and the exchange rate are given as follows:

$$\frac{dL_d(t,T)}{L_d(t,T)} = \gamma_{Ld}(t,T)\cdot\sigma_{Pd}(t,T+\delta)\,dt + \gamma_{Ld}(t,T)\cdot dW(t) \tag{2.6}$$

$$\frac{dL_f(t,T)}{L_f(t,T)} = \gamma_{Lf}(t,T)\cdot\left(\sigma_{Pf}(t,T+\delta)-\sigma_X(t)\right)dt + \gamma_{Lf}(t,T)\cdot dW(t) \tag{2.7}$$

$$\frac{dX(t)}{X(t)} = \left(r_d(t)-r_f(t)\right)dt + \sigma_X(t)\cdot dW(t) \tag{2.8}$$

where $t \in [0,T]$, $T \in [0,\tau]$ *and* $\sigma_{Pk}(t,T)$, $k \in \{d,f\}$ *is defined in (2.9).*

$$\sigma_{Pk}(t,T) = \begin{cases} \displaystyle\sum_{j=1}^{[\delta^{-1}(T-t)]} \frac{\delta L_k(t,T-j\delta)}{1+\delta L_k(t,T-j\delta)}\gamma_{Lk}(t,T-j\delta) & t\in[0,T-\delta] \\ & \&T-\delta>0, \\ 0 & \textit{otherwise.} \end{cases} \tag{2.9}$$

where $[\delta^{-1}(T-t)]$ denotes the greatest integer that is less than $\delta^{-1}(T-t)$.

When the domestic ZCB is used as the numeraire, the domestic forward probability measure Q^T is induced. The domestic forward measure Q^T can be defined by the Radon-Nikodym derivative $\dfrac{dQ^T}{dQ} = \dfrac{P_d(T,T)/P_d(t,T)}{\beta(T)/\beta(t)}$. From the Radon-Nikodym derivative, the relation of the Brownian motions under different measures can be shown as:

$$dW^Q(t) = dW^T(t) - \sigma_{Pd}(t,T)\,dt\,. \tag{2.10}$$

Substituting (2.10) into all the equations of Proposition 3.2, we can obtain the results presented by Proposition 2.3 below.

Proposition 2.3 THE CROSS-CURRENCY LIBOR MARKET MODEL UNDER THE DOMESTIC FORWARD MARTINGALE MEASURE

Under the domestic forward martingale measure Q^T, the processes of the forward LIBOR rates and the exchange rate are given as follows:

$$\frac{dL_d(t,T)}{L_d(t,T)}=\gamma_{Ld}(t,T)\cdot\left(\sigma_{Pd}(t,T+\delta)-\sigma_{Pd}(t,T)\right)dt+\gamma_{Ld}(t,T)\cdot dW(t) \tag{2.11}$$

$$\frac{dL_f(t,T)}{L_f(t,T)}=\gamma_{Lf}(t,T)\cdot\left(\sigma_{Pf}(t,T+\delta)-\sigma_{Pd}(t,T)-\sigma_X(t)\right)dt+\gamma_{Lf}(t,T)\cdot dW(t) \tag{2.12}$$

$$\frac{dX(t)}{X(t)}=\left(r_d(t)-r_f(t)-\sigma_X(t)\cdot\sigma_{Pd}(t,T)\right)dt+\sigma_X(t)\cdot dW(t) \tag{2.13}$$

where $t\in[0,T]$, $T\in[0,\tau]$ and $\sigma_{Pk}(t,T)$ is defined in (2.9).

Unlike the instantaneous forward rates in the HJM model, the forward LIBOR rates are market observable. Therefore, the volatility $\gamma_{Lk}(t,T)$ can be inversed from the market prices of the interest-rate derivatives traded in the market and $\sigma_{Pk}(t,T)$ can be calculated from (2.9). Because of the lognormal volatility structure, the forward LIBOR rates are almost surely positive, thereby preventing the negative rate problem in the Gaussian HJM model.

The cross-currency LIBOR market model is very general. It is useful for pricing many kinds of interest-rate derivatives. In Section 3, four variants of the cross-currency interest-rate exchange options are priced based on the cross-currency LIBOR market model.

3. Valuation of Quanto Interest-Rate Exchange Options

In this section, we derive the pricing formulae of four different types of quanto interest-rate exchange options (QIREOs) based on the cross-currency LIBOR market model. Introductions and analyses of each option are presented sequentially as follows.

3.1 Valuation of First-Type QIREOs

Definition 3.1 *A contingent claim with the payoff specified in (3.1.1) is called a First-Type QIREO* (Q_1IREO)

$$C_1(T) = N_d \omega \left[L_d^{\delta}(T,T) - L_f^{\eta}(T,T) \right]^+, \quad (3.1.1)$$

where

$L_d^{\delta}(T,T)$ = *the domestic T-matured LIBOR rates with a compounding period* δ

$L_f^{\eta}(T,T)$ = *the foreign T-matured LIBOR rates with a compounding period* η, $\eta \neq \delta$

N_d = *notional principal of the option, in units of domestic currency*

T = *the maturity date of the option*

$(x)^+$ = $Max(x,0)$

ω = *a binary operator (1 for a call option and -1 for a put option).*

An Q_1IREO is an option written on the difference between a domestic LIBOR rate with a compounding period δ and a foreign LIBOR rate with a compounding period η, but the final payments are denominated in domestic currency. In addition, an Q_1IREO with $\omega = 1$ represents a call option on the domestic LIBOR rate with the foreign LIBOR rate serving as the floating strike rate. On the contrary, an Q_1IREO with $\omega = -1$ denotes a put option with the foreign LIBOR rate as the underlying rate.

There are several benefits and applications associated with Q_1IREOs. First, Q_1IREOs provide a mechanism for taking advantage of cross-currency interest-rate differentials without directly incurring exchange rate risk. Second, investors can benefit from utilizing a corresponding Q_1IREO with making a correct assessment of the cross-currency interest-rate differential between two underlying LIBOR rates at some particular time point. Third, Q_1IREOs also can be used to provide added precision to strategies incorporating differential swaps. For example, a portfolio manager might use a differential swap to capitalize anticipated yield curve movements while also purchasing an Q_1IREO on the interest-rate differential in order to limit his downside risk. In addition, asset managers whose investments are mainly denominated in domestic currency can utilize Q_1IREOs to enhance portfolio return. A structure of this type can also be employed by liability managers and borrowers to effectively limit interest rate payments to the lower of either the domestic or foreign currency interest rates, without incurring exchange rate risk exposure.

Q_1IREO pricing is expressed in the following theorem, and the proof is provided in Appendix A.

Theorem 3.1 *The pricing formula of Q_1IREOs with the final payoff as specified in (3.1.1) is expressed as follows:*

$$C_1(t)=\omega N_d P_d(t,T)\left[L_d^{\delta}(t,T)e^{\int_t^T \overline{\mu}_{1d}^{\delta}(u,T,T+\delta)du}N(\omega d_{11})-L_f^{\eta}(t,T)e^{\int_t^T \overline{\mu}_{1f}^{\eta}(u,T,T+\eta)du}N(\omega d_{12})\right] \qquad (3.1.2)$$

where

$$d_{11}=\frac{\ln\left(\frac{L_d^{\delta}(t,T)}{L_f^{\eta}(t,T)}\right)+\int_t^T\left[\overline{\mu}_{1d}^{\delta}(u,T,T+\delta)-\overline{\mu}_{1f}^{\eta}(u,T,T+\eta)\right]du+\frac{1}{2}V_1^2}{V_1}$$

$$d_{12}=d_{11}-V_1$$

$$V_1^2=\int_t^T \lVert\gamma_{Ld}^{\delta}(u,T)-\gamma_{Lf}^{\eta}(u,T)\rVert^2\,du$$

$$\overline{\mu}_{1d}^{\delta}(t,T,T+\delta)=\gamma_{Ld}^{\delta}(t,T)\cdot\left[\overline{\sigma}_{P_d}^{s}(t,T+\delta)-\overline{\sigma}_{P_d}^{s}(t,T)\right]$$

$$\overline{\mu}_{1f}^{\eta}(t,T,T+\eta)=\gamma_{Lf}^{\eta}(t,T)\cdot\left[\overline{\sigma}_{P_f}^{s}(t,T+\eta)-\overline{\sigma}_{P_d}^{s}(t,T)-\sigma_x(t)\right]$$

$\omega=1$ (a call) or -1 (a put).

and $\overline{\sigma}_{Pk}^{s}(t,\cdot),\ k\in\{d,f\}$ is defined as (A.7) in Appendix A.

The pricing equation (3.1.2) may be regarded as a generalized representation of Margrabe (1978) in the framework of the cross-currency LMM. Note that when both compounding periods are identical ($\delta=\eta$), the pricing formula (3.1.2) reduces to the pricing model of a regular option on the spread between the domestic and the foreign LIBOR rates in the cross-currency LMM framework.

Theorem 3.1 not only provides the pricing formula for the Q_1IREOs but also reveals a clue to the construction of a hedging (replicating) portfolio for the Q_1IREOs.

For hedging, we rewrite equation (3.1.2) as equation (3.1.3) (the proof is provided in Appendix A) as follows

$$C_1(t)=\Delta_{1t}^{(1)}\left[P_d(t,T)-P_d(t,T+\delta)\right]-\Delta_{2t}^{(1)}\left[P_f(t,T)-P_f(t,T+\eta)\right], \qquad (3.1.3)$$

where

$$\Delta_{1t}^{(1)}=\omega N_d\left(1+\delta L_d^{\delta}(t,T)\right)\frac{1}{\delta}N(\omega d_{11})e^{\int_t^T \overline{\mu}_{1d}^{\delta}(u,T,T+\delta)du}$$

$$\Delta_{2t}^{(1)}=\omega N_d\left(1+\eta L_d^{\eta}(t,T)\right)\frac{1}{\eta}N(\omega d_{12})QA_1(t,T+\eta)$$

$$QA_1(t,T+\eta)=\frac{P_d(t,T+\eta)}{P_f(t,T+\eta)}\rho_1(t,T)$$

$$\rho_1(t,T)=e^{\int_t^T \overline{\mu}_{1f}^{\eta}(u,T,T+\eta)du}.$$

Equation (3.1.3) serves as a guide to the formation of a hedging portfolio $H_t^{(1)}$ for an

Q_1IREO. $H_t^{(1)}$ can be completed by a linear combination of four types of assets: holding long $\Delta_{1t}^{(1)}$ units of $P_d(t,T)$ and $\Delta_{2t}^{(1)}$ units of $P_f(t,T+\eta)$ and selling short $\Delta_{1t}^{(1)}$ units of $P_d(t,T+\delta)$ and $\Delta_{2t}^{(1)}$ units of $P_f(t,T)$.

The term $QA_1(t,T+\eta)$ appearing in (3.1.3) denotes the quanto adjustment due to the hedged risk of the exchange rate. This exchange rate adjustment is induced by the fact that expected foreign cash flow is derived under the domestic martingale measure, and by the compound correlations between all the involved factors (the domestic and foreign bonds and the exchange rate).

It is worth noting that the advantage of adopting the cross-currency BGM model rather than other traditional models is that all the parameters as shown in (3.1.1) and (3.1.2) can be easily obtained from market quotes, which makes the pricing formula more tractable and feasible for practitioners.

3.2 Valuation of Second-Type QIREOs

Definition 3.2 *A contingent claim with the payoff as specified in (3.2.1) is called a Second-Type QIREO* (Q_2IREO)

$$C_2(T) = \overline{X} N_f \left[L_f^{\delta}(T,T) - L_f^{\eta}(T,T) \right]^+, \tag{3.2.1}$$

where

N_f = *notional principal of the option, in units of foreign currency*

$\overline{X}$ = *the fixed exchange rate expressed as the domestic currency value of one unit of foreign currency.*

An Q_2IREO is an option written on the difference between two foreign LIBOR rates with different compounding periods δ and η , but the final payment is measured in domestic currency. From the viewpoint of domestic investors, holding an Q_2IREO acts in much the same way as longing a foreign yield-spread option, whose payoff is based on the difference between the two underlying foreign interest rates, denominated in foreign currency, and converting the foreign-currency payoff via multiplying the fixed exchange rate into the domestic-currency payoff.

Using Q_2IREOs has several benefits and applications. Domestic investors can benefit from utilizing a corresponding Q_2IREO with making a correct estimate of the differential between two foreign LIBOR rates at some particular time point, thereby avoiding exposure to exchange rate risk. For multinational enterprises or managers of cross-currency financial assets, Q_2IREOs can be used to enhance the interest profit of foreign assets or to reduce the interest cost arising from foreign liabilities without incurring exchange rate risk. Furthermore, Q_2IREOs can be used to limit the downside risks of some particular payments if a manager of cross-currency financial assets wants to manage the risk of foreign interest rate spread via a long-period foreign basis swap involving the exchange of two series of floating-rate cash flows in the same foreign currency.

The pricing formula of Q_2IREOs is expressed in Theorem 3.2 below and the proof is provided in Appendix B.

Theorem 3.2 *The pricing formula of Q_2IREOs with the final payoff as specified in (3.2.1) is presented as follows:*

$$C_2(t) = \overline{X} N_f P_d(t,T)\left[L_f^{\delta}(t,T)\, e^{\int_t^T \overline{\mu}_{2f}^{\delta}(u,T,T+\delta)du} N(d_{21}) - L_f^{\eta}(t,T)\, e^{\int_t^T \overline{\mu}_{2f}^{\eta}(u,T,T+\eta)du} N(d_{22}) \right] \quad (3.2.2)$$

where

$$d_{21} = \frac{\ln\left(\dfrac{L_f^{\delta}(t,T)}{L_f^{\eta}(t,T)}\right) + \int_t^T \left[\overline{\mu}_{2f}^{\delta}(u,T,T+\delta) - \overline{\mu}_{2f}^{\eta}(u,T,T+\eta) \right] du + \frac{1}{2} V_2^2}{V_2}$$

$$d_{22} = d_{21} - V_2$$

$$V_2^2 = \int_t^T \left\lVert \gamma_{Lf}^{\delta}(u,T) - \gamma_{Lf}^{\eta}(u,T) \right\rVert^2 du$$

$$\overline{\mu}_{2f}^{*}(t,T,T+*) = \gamma_{Lf}^{*}(t,T) \cdot \left[\overline{\sigma}_{P_f}^{s}(t,T+*) - \overline{\sigma}_{P_d}^{s}(t,T) - \sigma_x(t) \right], \quad * \in \{\delta, \eta\}.$$

Longstaff (1990), Fu (1996) and Miyazaki and Yoshida (1998) have derived the pricing formulae for interest rate difference options, which are written on the underlying difference between two domestic interest rates and denominated in domestic currency. In comparison with their pricing formulae, the major differences between Theorem 3.2 and their formulae lie in the fact that not only the "quanto-effect" is considered in Theorem 3.2, but also all parameters appearing in Theorem 3.2 can be extracted from market quotes, which makes our pricing formula more tractable and feasible for practitioners.

Once again, equation (3.2.2) can be written in terms of (3.2.3), and the proof is presented in Appendix B.

$$C_2(t) = \Delta_{1t}^{(2)}\left[P_f(t,T) - P_f(t,T+\delta) \right] - \Delta_{2t}^{(2)}\left[P_f(t,T) - P_f(t,T+\eta) \right], \quad (3.2.3)$$

where

$$\Delta_{1t}^{(2)} = \overline{X} N_f \left(1+\delta L_d^{\delta}\left(t,T\right)\right)\frac{1}{\delta} N\left(d_{21}\right) QA_2\left(t,T+\delta\right)$$

$$\Delta_{2t}^{(2)} = \overline{X} N_f \left(1+\eta L_d^{\eta}\left(t,T\right)\right)\frac{1}{\eta} N\left(d_{22}\right) QA_2\left(t,T+\eta\right)$$

$$QA_2\left(t,T+*\right) = \frac{P_d\left(t,T+*\right)}{P_f\left(t,T+*\right)} \rho_2^*\left(t,T\right), \quad * \in \left\{\delta,\eta\right\}$$

$$\rho_2^*\left(t,T\right) = e^{\int_t^T \overrightarrow{\mu}_{2f}^*\left(u,T,T+*\right)du}, \quad * \in \left\{\delta,\eta\right\}.$$

Equation (3.2.3) shows the composition of a hedging portfolio $H_t^{(2)}$ for an Q$_2$IREO: it holds long $\Delta_{1t}^{(2)}$ units of $P_f\left(t,T\right)$ and $\Delta_{2t}^{(2)}$ units of $P_f\left(t,T+\eta\right)$ and sells short $\Delta_{1t}^{(2)}$ units of $P_f\left(t,T+\delta\right)$ and $\Delta_{2t}^{(2)}$ units of $P_f\left(t,T\right)$. The implication of the quanto adjustment $QA_2\left(t,\cdot\right)$ is similar to $QA_1\left(t,T+\eta\right)$ as mentioned above.

3.3 Valuation of Third-Type QIREOs

Definition 3.3 *A contingent claim with the payoff as specified in (3.3.1) is called a Third-Type QIREO* (Q$_3$IREO)

$$C_3\left(T\right) = X\left(T\right) N_f \left[L_f^{\delta}\left(T,T\right) - L_f^{\eta}\left(T,T\right)\right]^+, \tag{3.3.1}$$

where

$X\left(T\right)$= *the floating exchange rate expressed as the domestic currency value of one unit of foreign currency at time T.*

An Q$_3$IREO is analogous to the Q$_2$IREO as specified in Subsection 3.2, but with the fixed exchange rate $\overline{X}$ replaced by the floating exchange rate $X\left(T\right)$ at maturity *T*. The structure of an Q$_3$IREO is slightly different from that of an Q$_2$IREO in that this option is directly affected by movements in the exchange rate. If the exchange rate moves upward, an investor using this option could enhance profits from the difference between both the foreign interest rates and the exchange rate. And a seller of this option could reduce payments due to downward movements in a foreign currency's value.

Since the Q$_3$IREO can be priced in a similar way as the Q$_2$IREO, we omit the proof. The result is available upon request from the authors.

Theorem 3.3 *The pricing formula of Q$_3$IROs with the final payoff as expressed in (3.3.1) is presented as follows:*

$$C_3(t) = X(t) N_f P_f(t,T)\left[L_f^\delta(t,T) e^{\int_t^T\left[\overline{\mu_3}_f^\delta(u,T,T+\delta)\right]du} N(d_{31}) - L_f^\eta(t,T) e^{\int_t^T\left[\overline{\mu_3}_f^\eta(u,T,T+\eta)\right]du} N(d_{32})\right], \qquad (3.3.2)$$

where

$$d_{31} = \frac{\ln\left(\dfrac{L_f^\delta(t,T)}{L_f^\eta(t,T)}\right) + \int_t^T\left[\overline{\mu_3}_f^\delta(u,T,T+\delta) - \overline{\mu_3}_f^\eta(u,T,T+\eta)\right]du + \frac{1}{2}V_3^2}{V_3}$$

$$d_{32} = d_{31} - V_3$$

$$V_3^2 = \int_t^T \left\| \gamma_{Lf}^\delta(u,T) - \gamma_{Lf}^\eta(u,T) \right\|^2 du$$

$$\overline{\mu_3}_f^*(t,T,T+*) = \gamma_{Lf}^*(t,T)\cdot\left[\overline{\sigma}_{P_f}^s(t,T+*) - \overline{\sigma}_{P_f}^s(t,T)\right],\ * \in \{\delta,\eta\}.$$

Similarly, we rewrite (3.3.2) to obtain (3.3.3) as follows

$$C_3(t) = \Delta_{1t}^{(3)}\left[P_f(t,T) - P_f(t,T+\delta)\right] - \Delta_{2t}^{(3)}\left[P_f(t,T) - P_f(t,T+\eta)\right], \qquad (3.3.3)$$

where

$$\Delta_{1t}^{(3)} = X(t) N_f \left(1 + \delta L_f^\delta(t,T)\right)\frac{1}{\delta} e^{\int_t^T \overline{\mu}_{3f}^\delta(u,T,T+\delta)du} N(d_{31})$$

$$\Delta_{2t}^{(3)} = X(t) N_f \left(1 + \eta L_f^\eta(t,T)\right)\frac{1}{\eta} e^{\int_t^T \overline{\mu}_{3f}^\eta(u,T,T+\eta)du} N(d_{32}).$$

Equation (3.3.3) also implies a composition for a hedging portfolio $H_t^{(3)}$ similar to that given in the previous theorems. It is worth noting that the quanto adjustment disappears in (3.3.3), since the exchange rate risk in the Q_3IREO is unhedged; this option is directly affected by unanticipated changes in the exchange rate.

3.4 Valuation of Fourth-Type QIREOs

Definition 3.4 *A contingent claim with the payoff as specified in (3.4.1) is called a Fourth-Type QIREO* (Q_4IREO)

$$C_4(T) = \omega\left[X(T) N_f L_f^\delta(T,T) - N_d L_d^\eta(T,T)\right]^+. \qquad (3.4.1)$$

ω = *a binary operator (1 for a call option and -1 for a put option).*

An Q_4IREO is an option written on the difference between a foreign interest payment based on the foreign LIBOR rate with a compounding period δ and a domestic interest payment based on the domestic LIBOR rate with a compounding period η.

This option is slightly different from those options described in the above subsections. It can be considered as an option to exchange domestic-currency-denominated interest payments for foreign-currency-denominated interest payments.

Theorem 3.4 below presents the pricing formula of an Q$_4$IREO. Its proof follows in a similar way as the previous options. The result is available upon request from the authors.

Theorem 3.4 *The pricing formula of Q$_4$IREOs with the final payoff as expressed in (3.4.1) is presented as follows:*

$$\begin{aligned} C_4(t) &= \omega X(t) N_f P_f(t,T) L_f^{\delta}(t,T) e^{\int_t^T \left[\overline{\mu_{4\,g}^{\delta}}(u,T,T+\delta)\right] du} N(\omega d_{41}) \\ &\quad -\omega N_d P_d(t,T) L_d^{\eta}(t,T) e^{\int_t^T \left[\overline{\mu_{4\,d}^{\eta}}(u,T,T+\eta)\right] du} N(\omega d_{42}) \end{aligned} \tag{3.4.2}$$

where

$$d_{41} = \frac{\ln\left(\dfrac{X(t) N_f P_f(t,T) L_f^{\delta}(t,T)}{N_d P_d(t,T) L_d^{\eta}(t,T)}\right) + \int_t^T \left[\overline{\mu_{4\,g}^{\delta}}(u,T,T+\delta) - \overline{\mu_{4\,d}^{\eta}}(u,T,T+\eta)\right] du + \frac{1}{2} V_4^2}{V_4}$$

$$d_{42} = d_{41} - V_4$$

$$V_4^2 = \int_t^T \left\| \gamma_g^{\delta}(u,T) - \gamma_{Ld}^{\eta}(u,T) \right\|^2 du$$

$$\overline{\mu_{4\,g}^{\delta}}(t,T,T+\delta) = \gamma_{Lf}^{\delta}(t,T) \cdot \left[\overline{\sigma_{P_f}^{s}}(t,T+\delta) - \overline{\sigma_{P_f}^{s}}(t,T)\right]$$

$$\overline{\mu_{4\,d}^{\eta}}(t,T,T+\eta) = \gamma_{Ld}^{\eta}(t,T) \cdot \left[\overline{\sigma_{P_d}^{s}}(t,T+\eta) - \overline{\sigma_{P_d}^{s}}(t,T)\right].$$

In order to obtain a hedging portfolio, equation (3.4.2) is rewritten as equation (3.4.3).

$$C_4(t) = \Delta_{1t}^{(4)} \left[P_f(t,T) - P_f(t,T+\delta)\right] - \Delta_{2t}^{(4)} \left[P_d(t,T) - P_d(t,T+\eta)\right], \tag{3.4.3}$$

where

$$\Delta_{1t}^{(4)} = \omega X(t) N_f \left(1 + \delta L_f^{\delta}(t,T)\right) \frac{1}{\delta} e^{\int_t^T \overline{\mu_{4\,g}^{\delta}}(u,T,T+\delta) du} N(\omega d_{41})$$

$$\Delta_{2t}^{(4)} = \omega N_d \left(1 + \eta L_d^{\eta}(t,T)\right) \frac{1}{\eta} e^{\int_t^T \overline{\mu_{4\,d}^{\eta}}(u,T,T+\eta) du} N(\omega d_{42}).$$

Equation (3.4.3) shows the composition of a hedging portfolio $H_t^{(4)}$ for an Q$_4$IREO: holding long $\Delta_{1t}^{(4)}$ units of $P_f(t,T)$ and $\Delta_{2t}^{(4)}$ units of $P_d(t,T+\eta)$ and selling short $\Delta_{1t}^{(4)}$ units of $P_f(t,T+\delta)$ and $\Delta_{2t}^{(4)}$ units of $P_d(t,T)$. Due to the unhedged exchange-rate risk inherent in the Q$_4$IREO, the quanto adjustment does not exist in equation (3.4.3) as in the case examined in Subsection 3.3; this option is directly affected by exchange-rate movements as well.

In Section 4, we provide a calibration procedure and numerical examples showing the accuracy of the pricing formulae.

4. Calibration Procedure and Numerical Examples

In this section, we first provide a calibration procedure and then examine the accuracy of the pricing formula via a comparison with Monte Carlo simulation.

4.1 Calibration Procedure

With the advantage of the pricing formulae for caps and floors which are consistent with the popular Black formula [1976], the cross-currency LIBOR market model is easier for calibration. We employ the mechanism presented by Rebonato [1999] to engage in a simultaneous calibration of the cross-currency LIBOR market model to the percentage volatilities and the correlation matrix of the underlying forward LIBOR rates and the exchange rate.

Assume that there are n domestic forward LIBOR rates, n foreign forward LIBOR rates and an exchange rate in an m-factor framework. The steps to calibrate the model parameters are presented as follows:

First, as given in Brigo and Mercurio [2001], we assume that the domestic forward LIBOR rate, $L_d\left(t,\cdot\right)$, has a piecewise-constant instantaneous total volatility structure depending only on the time-to-maturity (i.e., $V_{i,j}^d = \upsilon_{i-j}^d$). The elements in Exhibit 1, which specify the instantaneous total volatility applied to each period for each rate, can be stripped from market data. A detailed computational process is presented in Hull [2003].

The case of the foreign forward LIBOR rate, $L_f\left(t,\cdot\right)$, can be carried out in a way similar to the domestic case. In addition, we also assume that the exchange rate $X\left(t\right)$ has a piecewise-constant instantaneous total volatility structure. The elements in Exhibit 2, which represent the instantaneous total volatility applied to each period for the exchange rate, can be calculated from the prices of the on-the-run options in the market. However, the durations of the exchange rate options are usually shorter than one year, so the market-obtainable elements in Exhibit 2 are usually not sufficient for pricing interest options. This problem may be resolved by using the implied (or historical) volatility of the underlying exchange rate, and assuming that the term structure of volatilities is flat (i.e., $\zeta_X(t) = \zeta_X$ for $t \in (t_0, t_n]$).

Exhibit 1: Instantaneous Volatilities of $\{L_k(t,\cdot)\}_{k\in\{d,f\}}$

Instant. Total Vol.	Time $t \in (t_0, t_1]$	$(t_1, t_2]$	$(t_2, t_3]$	$\cdots$	$(t_{n-2}, t_{n-1}]$
Fwd. Rate: $L_k(t, t_1)$	$V_{1,1}^k = \upsilon_0^k$	Dead	Dead	$\cdots$	Dead
$L_k(t, t_2)$	$V_{2,1}^k = \upsilon_1^k$	$V_{2,2}^k = \upsilon_0^k$	Dead	$\cdots$	Dead
$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$L_k(t, t_{n-1})$	$V_{n-1,1}^k = \upsilon_{n-2}^k$	$V_{n-1,2}^k = \upsilon_{n-3}^k$	$V_{n-1,3}^k = \upsilon_{n-4}^k$	$\cdots$	$V_{n-1,n-1}^k = \upsilon_0^k$

Exhibit 2 : Instantaneous Volatilities of the Exchange Rate

Instant. Total Vol.	Time $t \in (t_0, t_1]$	$(t_1, t_2]$	$(t_2, t_3]$	$\cdots$	$(t_{n-2}, t_{n-1}]$
Fwd. Rate: $X(t)$	$V_{X1} = \zeta_1$	$V_{X2} = \zeta_2$	$V_{X3} = \zeta_3$	$\cdots$	$V_{Xn} = \zeta_n$

Second, we use the historical price data of the domestic and foreign forward LIBOR rates and the exchange rate to derive a full-rank $(2n+1)\times(2n+1)$ instantaneous-correlation matrix Γ. Thus, Γ is a positive-definite symmetric matrix and can be written as

$$\Gamma = H \Lambda H'$$

where H is a real orthogonal matrix and Λ is a diagonal matrix. Let $A \equiv H\Lambda^{1/2}$ and thus $AA' = \Gamma$. In this way, we can find a suitable m-rank matrix B such that the m-rank matrix • $\Gamma^B = BB'$ can be used to mimic the market correlation matrix Γ, where $m \leq$• $2n+1$.

The purpose of the second step is to replace the $2n+1$-dimensional original Brownian motions $dW(t)$ with $BdZ(t)$, where $dZ(t)$ is a vector of m-dimensional Brownian motions. In other words, we change the market correlation structure

$$dW(t)dW(t)' = \Gamma dt$$

to a modeled correlation structure

$$BdZ(t)\left(BdZ(t)\right)' = BdZ(t)dZ(t)'B' = BB'dt = \Gamma^B dt$$

The remaining problem is how to choose a suitable matrix B. Rebonato [1999] proposed the following form for the ith row of B:

$$b_{i,k}=\begin{cases}\cos\theta_{i,k}\Pi_{j=1}^{k-1}\sin\theta_{i,j} & if\ \ k=1,2,...,m-1\\ \Pi_{j=1}^{k-1}\sin\theta_{i,j} & if\ \ k=m\end{cases}$$

for $i=1,2,...,2n+1$. By finding a $\hat{\theta}$ that solves the optimization problem

$$\min_{\theta}\sum_{i,j=1}^{n}\left|\Gamma_{i,j}^{B}-\Gamma_{i,j}\right|^{2}$$

and substituting $\hat{\theta}$ into B, we obtain a suitable matrix $\tilde{B}$ such that $\Gamma^{B}\left(=\tilde{B}\tilde{B}'\right)$ is an approximate correlation matrix for Γ.

Third, $\tilde{B}$ can be used to distribute the instantaneous total volatility to each Brownian motion at each period for the exchange rate and to each LIBOR rate without changing the amount of the instantaneous total volatility. That is,

$$V_{i,j}^{k}\left(\tilde{B}(i,1),\tilde{B}(i,2),...,\tilde{B}(i,m)\right)=\left(\gamma_{Lk1}(t,t_i),\gamma_{Lk2}(t,t_i),...,\gamma_{Lkm}(t,t_i)\right),$$

$$\zeta_{j}\left(\tilde{B}(n,1),\tilde{B}(n,2),...,\tilde{B}(n,m)\right)=\left(\sigma_{X1}(t),\sigma_{X2}(t),...,\sigma_{Xm}(t)\right),$$

where $i=1,2,...,n-1$ and $t\in(t_{j-1},t_j]$, for each $j=1,2,...,n$.

Under the assumption that the instantaneous total volatility structures are piecewise-constant, the previous procedure represents a general calibration method without a constraint on choosing the number of factors. Via the distributing matrix $\tilde{B}$, the individual instantaneous volatility applied to each Brownian motion at each period for each process can be derived. With these data calibrated from the market correlation matrix and volatilities, we can employ Monte Carlo simulation to price any associated interest rate derivatives. The data can also be used to calculate the prices of the QIREOs as derived in Theorems 1, 2, 3 and 4.

4.2 Numerical Analysis

This subsection offers some practical examples that examine the accuracy of the pricing formulas as derived in the previous section and compare the results with Monte Carlo simulation. Based on actual market data, as shown in Exhibits 5 to 10 in Appendix C, the 1-year and 3-year Q_2IREOs with $\delta=\eta=\frac{1}{2}$ year and $\omega=-1$ in Theorem 1 are priced at different semiannually dates, and the results are listed in Exhibits 3 and 4. The flat volatility of the exchange rate is assumed to be 20%. The notional value is assumed to be \$1. The simulation is based on 50,000 sample paths. Note that in the examples, the domestic country is the U.S. and the foreign country is the U.K. By comparison to Monte Carlo simulation, the pricing formulas have shown to be accurate and robust for the recent market data. The empirical examples associated with the other three theorems have also shown satisfactory accuracy.[6]

Exhibit 3: The 1-Year Q_1IREO

Date	2006/1/2	2006/7/3	2007/1/1	2007/7/2
Thm 1	1.2683×10^{-3}	1.2802×10^{-3}	5.0714×10^{-3}	9.5540×10^{-3}
M.C.	1.2682×10^{-3}	1.2811×10^{-3}	5.0779×10^{-3}	9.5594×10^{-3}
s.e.	1.2812×10^{-5}	1.2560×10^{-5}	2.3516×10^{-5}	2.8212×10^{-5}

The prices of a 1-year Q_1IRO with semiannual accrual periods are presented in this exhibit. The abbreviations M.C. and s.e. represent the results of Monte Carlo simulations and their standard errors, respectively.

Exhibit 4: The 3-Year Q_1IREO

Date	2006/1/2	2006/7/3	2007/1/1	2007/7/2
Thm 1	4.0575×10^{-3}	3.1143×10^{-3}	5.6667×10^{-3}	7.3662×10^{-3}
M.C.	4.0546×10^{-3}	3.1229×10^{-3}	5.6588×10^{-3}	7.3560×10^{-3}
s.e.	7.4232×10^{-5}	6.3378×10^{-5}	8.5226×10^{-5}	9.9884×10^{-5}

The prices of a 3-year Q_1IRO with semiannual accrual periods are presented in this exhibit.

5. Conclusions

We have adopted a general cross-currency LIBOR market model to price four different types of QIREOs with four theorems. The derived pricing formulae represent the general formulae of Margrabe (1978) in the framework of the cross-currency LMM, and are familiar to practitioners for easy practical implementation. These pricing formulae have been examined to be very accurate as compared with Monte-Carlo simulation.

Moreover, we have provided the hedging strategies for the QIREOs via the pricing formulae and discussed the calibration procedure in detail. Since the LIBOR rate is market observable and its related derivatives, such as caps and swaptions, are actively traded in the markets, it is easier to calibrate these model parameters than with traditional interest-rate models. Thus, the QIREO-pricing formulae derived under the cross-currency LIBOR market model are more tractable and feasible for practical implementation.

Appendix A: Proof of Theorem 3.1

A.1 Proof of Equation (3.1.2)

By applying the martingale pricing method, the price of an Q_1IRO at time t, $0 \le t \le T$, is derived as follows:

$$C_1(t) = N_d E^Q \left\{ e^{\left(-\int_t^T r_d(s)ds\right)} \omega\left[L_d^\delta(T,T) - L_f^\eta(T,T)\right]^+ \middle| F_t \right\} \quad \text{(A.1)}$$

$$= N_d E^Q \left\{ \frac{P_d(T,T)/P_d(t,T)}{\beta_d(T)/\beta_d(t)} P_d(t,T) \omega\left[L_d^\delta(T,T) - L_f^\eta(T,T)\right]^+ \middle| F_t \right\} \quad \text{(A.2)}$$

$$= N_d P_d(t,T) E^T \left\{ \omega\left[L_d^\delta(T,T) - L_f^\eta(T,T)\right] \mathrm{I}_A \middle| F_t \right\}, \quad A = \left\{ \omega\left[L_d^\delta(T,T) - L_f^\eta(T,T)\right] > 0 \right\} \quad \text{(A.3)}$$

$$= \omega N_d P_d(t,T) \underbrace{E^T \left\{ L_d^\delta(T,T) \mathrm{I}_A \middle| F_t \right\}}_{(A-I)} - \omega N_d P_d(t,T) \underbrace{E^T \left\{ L_f^\eta(T,T) \mathrm{I}_A \middle| F_t \right\}}_{(A-II)} \quad \text{(A.4)}$$

where

$E^Q(\cdot)$ denotes the expectation under the domestic martingale measure Q.

$E^T(\cdot)$ denotes the expectation under the domestic forward martingale measure Q^T defined by the Radon-Nikodym derivative $\dfrac{dQ^T}{dQ} = \dfrac{P_d(T,T)/P_d(t,T)}{\beta_d(T)/\beta_d(t)}$.

ω is a binary operator (1 for a call option and -1 for a put option).

I_A is an indicator function with $\begin{cases} 1, & if \quad \omega\left[L_d^\delta(T,T) - L_f^\eta(T,T)\right] > 0 \\ 0, & otherwise \end{cases}$.

Part (A-I) and (A-II) are solved, respectively, as follows.

From Proposition 2.3, the dynamics of $L_d^\delta(t,T)$ and $L_f^\eta(t,T)$ under the domestic forward measure Q^T are shown as follows:

$$\frac{dL_d^\delta(t,T)}{L_d^\delta(t,T)} = \gamma_{Ld}^\delta(t,T) \cdot \left[\sigma_{P_d}(t,T+\delta) - \sigma_{P_d}(t,T)\right] dt + \gamma_{Ld}^\delta(t,T) \cdot dW_t^T, \quad \text{(A.5)}$$

$$\frac{dL_f^\eta(t,T)}{L_f^\eta(t,T)} = \gamma_{Lf}^\eta(t,T) \cdot \left[\sigma_{P_f}(t,T+\eta) - \sigma_{P_d}(t,T) - \sigma_x(t)\right] dt + \gamma_{Lf}^\eta(t,T) \cdot dW_t^T. \quad \text{(A.6)}$$

According to the definition of the bond volatility process $\{\sigma_{Pk}(t,T)\}_{t\in[s,T]}$ in (2.9), $\{\sigma_{Pk}(t,T)\}_{t\in[s,T]}$ is not deterministic. Thus, the stochastic differential equations (A.5) and

(A.6) are not allowed to solve the distributions of $L_d^\delta(T,T)$ and $L_f^\eta(T,T)$. We can, however, approximate $\sigma_{Pk}(t,T)$ by $\overline{\sigma}^s_{Pk}(t,T)$ which is defined by:

$$\overline{\sigma}^s_{Pk}(t,T)=\begin{cases}\displaystyle\sum_{j=1}^{[\delta^{-1}(T-t)]}\frac{\delta L_k(s,T-j\delta)}{1+\delta L_k(s,T-j\delta)}\gamma_{Lk}(t,T-j\delta) & t\in[0,T-\delta] \\ & \& \ T-\delta>0, \\ 0 & otherwise.\end{cases} \tag{A.7}$$

where $0\le s\le t\le T$ and $k\in\{d,f\}$. Accordingly, the calendar time of the process $\{L_k(t,T)\}_{t\in[s,T]}$ in (A.7) is frozen at its initial time s, thus the process $\left\{\overline{\sigma}^s_{Pk}(t,T)\right\}_{t\in[s,T]}$ becomes deterministic. This is the Wiener chaos order 0 approximation, which is first used for pricing swaptions by BGM (1997). It was further developed in Brace, Dun and Barton (1998) and formalized by Brace and Womersley (2000).

Substituting $\overline{\sigma}^s_{Pk}(t,T)$ for $\sigma_{Pk}(t,T)$ in the drift terms of (A.5) and (A.6), we obtain:

$$\frac{dL_d^\delta(t,T)}{L_d^\delta(t,T)}=\gamma_{Ld}^\delta(t,T)\cdot\left[\overline{\sigma}^s_{P_d}(t,T+\delta)-\overline{\sigma}^s_{Pd}(t,T)\right]dt+\gamma_{Ld}^\delta(t,T)\cdot dW_t^T, \tag{A.8}$$

$$\frac{dL_f^\eta(t,T)}{L_f^\eta(t,T)}=\gamma_{Lf}^\eta(t,T)\cdot\left[\overline{\sigma}^s_{P_f}(t,T+\eta)-\overline{\sigma}^s_{P_d}(t,T)-\sigma_x(t)\right]dt+\gamma_{Lf}^\eta(t,T)\cdot dW_t^T. \tag{A.9}$$

In this way, the drift and volatility terms in (A.8) and (A.9) are deterministic. Therefore, we can solve (A.8) and (A.9) and find the approximate distributions of $L_d^\delta(T,T)$ and $L_f^\eta(T,T)$.

Solving the stochastic differential equations(A.8) and (A.9), we obtain:

$$L_d^\delta(T,T)=L_d^\delta(t,T)e^{\int_t^T\left[\overline{\mu}^\delta_{1d}(u,T,T+\delta)-\frac{1}{2}\|\gamma_{Ld}^\delta(u,T)\|^2\right]du+\int_t^T\gamma_{Ld}^\delta(u,T)\cdot dW_u^T}, \tag{A.10}$$

$$L_f^\eta(T,T)=L_f^\eta(t,T)e^{\int_t^T\left[\overline{\mu}^\eta_{1f}(u,T,T+\eta)-\frac{1}{2}\|\gamma_{Lf}^\eta(u,T)\|^2\right]du+\int_t^T\gamma_{Lf}^\eta(u,T)dW_u^T} \tag{A.11}$$

where

$$\overline{\mu}^\delta_{1d}(u,T,T+\delta)=\gamma_{Ld}^\delta(u,T)\cdot\left[\overline{\sigma}^s_{P_d}(u,T+\delta)-\overline{\sigma}^s_{P_d}(u,T)\right], \tag{A.12}$$

$$\overline{\mu}^\eta_{1f}(u,T,T+\eta)=\gamma_{Lf}^\eta(u,T)\cdot\left[\overline{\sigma}^s_{P_f}(u,T+\eta)-\overline{\sigma}^s_{P_d}(u,T)-\sigma_x(u)\right]. \tag{A.13}$$

By substituting (A.10) into (A-I), (A-I) can be rewritten as:

$$(A\text{-}I)=L_d^\delta(t,T)e^{\int_t^T\overline{\mu}^\delta_{1d}(u,T,T+\delta)du}E^T\left\{e^{-\frac{1}{2}\int_t^T\|\gamma_{Ld}^\delta(u,T)\|^2du+\int_t^T\gamma_{Ld}^\delta(u,T)\cdot dW_u^T}\mathrm{I}_A\middle|F_t\right\} \tag{A.14}$$

$$= L_d^{\delta}(t,T) e^{\int_t^{T} \overline{\mu}_{1d}^{\delta}(u,T,T+\delta) du} P_r^{R_1}\left\{\omega\left[L_d^{\delta}(T,T) - L_f^{\eta}(T,T)\right] \geq 0 \middle| F_t\right\}. \quad \text{(A.15)}$$

$P_r^{R_1}[\cdot]$ denotes the probability measured in the martingale measure R_1 which is defined by the Radon-Nikodym derivative $\frac{dR_1}{dQ^T} = e^{-\frac{1}{2}\int_t^T \|\gamma_{Ld}^{\delta}(u,T)\|^2 du + \int_t^T \gamma_{Ld}^{\delta}(u,T)\cdot dW_u^T}$.

From the Radon Nikodym derivative $\frac{dR_1}{dQ^T}$, we know that

$$dW_t^{R_1} = dW_t^T - \gamma_{Ld}^{\delta}(t,T) dt. \quad \text{(A.16)}$$

Under the measure R_1, we obtain the results by substituting (A.16) into (A.10) and (A.11):

$$L_d^{\delta}(T,T) = L_d^{\delta}(t,T) e^{\int_t^T \left[\overline{\mu}_{1d}^{\delta}(u,T,T+\delta) + \frac{1}{2}\|\gamma_{Ld}^{\delta}(u,T)\|^2\right] du + \int_t^T \gamma_{Ld}^{\delta}(u,T)\cdot dW_u^{R_1}}, \quad \text{(A.17)}$$

$$L_f^{\eta}(T,T) = L_f^{\eta}(t,T) e^{\int_t^T \left[\overline{\mu}_f^{\eta}(u,T,T+\eta) - \frac{1}{2}\|\gamma_{Lf}^{\eta}(u,T)\|^2 + \gamma_{Ld}^{\delta}(u,T)\cdot\gamma_{Lf}^{\eta}(u,T)\right] du + \int_t^T \gamma_{Lf}^{\eta}(u,T)\cdot dW_u^{R_1}}. \quad \text{(A.18)}$$

By inserting (A.17) and (A.18) into $P_r^{R_1}[\cdot]$, the probability can be obtained after rearrangement as follows:

$$P_r^{R_1}\left\{\omega\left[L_d^{\delta}(T,T) - L_f^{\eta}(T,T) \geq 0 \middle| F_t\right]\right\} = N(\omega d_{11}) \quad \text{(A.19)}$$

where

$N(\cdot)$ represents the cumulative density function of the normal distribution,

$$d_{11} = \frac{\ln\left(\frac{L_d^{\delta}(t,T)}{L_f^{\eta}(t,T)}\right) + \int_t^T \left[\overline{\mu}_{1d}^{\delta}(u,T,T+\delta) - \overline{\mu}_{1f}^{\eta}(u,T,T+\eta)\right] du + \frac{1}{2}V_1^2}{V_1}, \quad \text{(A.20)}$$

$$V_1^2 = \int_t^T \|\gamma_{Ld}^{\delta}(u,T) - \gamma_{Lf}^{\eta}(u,T)\|^2 du. \quad \text{(A.21)}$$

The procedures to solve (A-II) are similar to those of (A-I).

By substituting (A.11) into (A-II), (A-II) is derived as follows:

$$(A\text{-}II) = L_f^{\eta}(t,T) e^{\int_t^T \overline{\mu}_{1f}^{\eta}(u,T,T+\eta) du} E^T\left\{e^{-\frac{1}{2}\int_t^T \|\gamma_{Lf}^{\eta}(u,T)\|^2 du + \int_t^T \gamma_{Lf}^{\eta}(u,T)\cdot dW_u^T} \mathrm{I}_A \middle| F_t\right\} \quad \text{(A.22)}$$

$$= L_f^{\eta}(t,T) e^{\int_t^T \overline{\mu}_{1f}^{\eta}(u,T,T+\eta) du} P_r^{R_2}\left\{\omega\left[L_d^{\delta}(T,T) - L_f^{\eta}(T,T) \geq 0 \middle| F_t\right]\right\}. \quad \text{(A.23)}$$

$P_r^{R_2}[\cdot]$ denotes the probability measured in the martingale measure R_2 which is defined by the Radon-Nikodym derivative $\frac{dR_2}{dQ^T} = e^{-\frac{1}{2}\int_t^T \|\gamma_{Lf}^{\eta}(u,T)\|^2 du + \int_t^T \gamma_{Lf}^{\eta}(u,T)\cdot dW_u^T}$.

From the Radon-Nikodym derivative $\frac{dR_2}{dQ^T}$, we find that

$$dW_t^{R_2} = dW_t^T - \gamma_{Lf}^{\eta}(t,T)dt . \tag{A.24}$$

Under the measure R_2, we obtain the results by substituting (A.24) into (A.10) and (A.11):

$$L_d^{\delta}(T,T) = L_d^{\delta}(t,T) e^{\int_t^T \left[\overline{\mu}_{1d}^{\delta}(u,T,T+\delta) - \frac{1}{2}\left\| \gamma_{Ld}^{\delta}(u,T) \right\|^2 + \gamma_{Ld}^{\delta}(u,T)\cdot\gamma_{Lf}^{\eta}(u,T) \right] du + \int_t^T \gamma_{Ld}^{\delta}(u,T)\cdot dW_u^{R_2}} , \tag{A.25}$$

$$L_f^{\eta}(T,T) = L_f^{\eta}(t,T) e^{\int_t^T \left[\overline{\mu}_{f}^{\eta}(u,T,T+\eta) + \frac{1}{2}\left\| \gamma_{Lf}^{\eta}(u,T) \right\|^2 \right] du + \int_t^T \gamma_{Lf}^{\eta}(u,T)\cdot dW_u^{R_2}} . \tag{A.26}$$

Inserting (A.25) and (A.26) into $P_r^{R_2}[\cdot]$, we obtain

$$P_r^{R_2}\left\{ \omega\left[L_d^{\delta}(T,T) - L_f^{\eta}(T,T) \geq 0 \middle| F_t \right] \right\} = N(\omega d_{12}) \tag{A.27}$$

$$d_{12} = d_{11} - V_1 . \tag{A.28}$$

By combining A(4), A(15), A(19), A(23) with A(27), equation (3.1.2) of Theorem 3.1 is obtained.

A.2 Proof of Equation (3.1.3)

By definition,

$$L_d^{\delta}(t,T) = \frac{1}{\delta}\left(\frac{P_d(t,T)}{P_d(t,T+\delta)} - 1 \right) \tag{A.29}$$

$$L_f^{\eta}(t,T) = \frac{1}{\eta}\left(\frac{P_f(t,T)}{P_f(t,T+\eta)} - 1 \right) \tag{A.30}$$

By substituting (A.29) and (A.30) into (3.1.2) and rearranging it, equation (3.1.3) is derived.

Appendix B: Proof of Theorem 3.2

B.1 Proof of Equation (3.2.2)

The pricing formula of an Q2IRO at time t, $0 \le t \le T$, is derived as follows:

$$C_2(t) = E^Q\left\{ e^{\left(-\int_t^T r_d(s)ds\right)} N_f \overline{X}\left[L_f^\delta(T,T) - L_f^\eta(T,T)\right]^+ \middle| F_t \right\} \tag{B.1}$$

$$= N_f \overline{X} P_d(t,T) E^T\left\{ \left[L_f^\delta(T,T) - L_f^\eta(T,T)\right] \mathrm{I}_A \middle| F_t \right\}, \quad A = \left\{L_f^\delta(T,T) > L_f^\eta(T,T)\right\} \tag{B.2}$$

$$= N_f \overline{X} P_d(t,T) \underbrace{E^T\left\{ L_f^\delta(T,T) \mathrm{I}_A \middle| F_t \right\}}_{(B-I)} - N_f \overline{X} P_d(t,T) \underbrace{E^T\left\{ L_f^\eta(T,T) \mathrm{I}_A \middle| F_t \right\}}_{(B-II)} \tag{B.3}$$

Parts (B-I) and (B-II) are then solved respectively.

From Proposition 2.3, the dynamics of $L_f^\delta(t,T)$ and $L_f^\eta(t,T)$ under the domestic forward measure Q^T are listed as follows:

$$\frac{dL_f^*(t,T)}{L_f^*(t,T)} = \gamma_{Lf}^*(t,T) \cdot \left[\sigma_{P_f}(t,T+*) - \sigma_{P_d}(t,T) - \sigma_x(t)\right] dt + \gamma_{Lf}^*(t,T) \cdot dW_t^T, \ * \in \{\delta, \eta\} \tag{B.4}$$

Substituting $\overline{\sigma}_{Pk}^{s}(t,\cdot)$ as defined in (A.7) for $\sigma_{Pk}(t,\cdot)$ in (B.4), we get

$$\frac{dL_f^*(t,T)}{L_f^*(t,T)} = \gamma_{Lf}^*(t,T) \cdot \left[\overline{\sigma}_{P_f}^{s}(t,T+*) - \overline{\sigma}_{P_d}^{s}(t,T) - \sigma_x(t)\right] dt + \gamma_{Lf}^*(t,T) \cdot dW_t^T, \ * \in \{\delta, \eta\} \tag{B.5}$$

By solving the stochastic differential equation (B.5), we obtain

$$L_f^*(T,T) = L_f^*(t,T) e^{\int_t^T \left[\overline{\mu}_{2f}^*(u,T,T+*) - \frac{1}{2}\|\gamma_{Lf}^*(u,T)\|^2\right] du + \int_t^T \gamma_{Lf}^*(u,T) dW_u^T}, \qquad * \in \{\delta, \eta\} \tag{B.6}$$

where

$$\overline{\mu}_{2f}^*(u,T,T+*) = \gamma_{Lf}^*(u,T) \cdot \left[\overline{\sigma}_{P_f}^{s}(t,T+*) - \overline{\sigma}_{P_d}^{s}(t,T) - \sigma_x(u)\right], \qquad * \in \{\delta, \eta\}. \tag{B.7}$$

(B.6) is substituted into (B-I) to derive (B-I) as follows:

$$(B\text{-}I) = L_f^\delta(t,T) e^{\int_t^T \overline{\mu}_{2f}^\delta(u,T,T+\delta) du} E^T\left\{ e^{-\frac{1}{2}\int_t^T \|\gamma_{Lf}^\delta(u,T)\|^2 du + \int_t^T \gamma_{Lf}^\delta(u,T) \cdot dW_u^T} \mathrm{I}_A \middle| F_t \right\} \tag{B.8}$$

$$= L_f^\delta(t,T) e^{\int_t^T \overline{\mu}_{2f}^\delta(u,T,T+\delta) du} P_r^{R_1}\left[L_f^\delta(T,T) \ge L_f^\eta(T,T) \middle| F_t \right]. \tag{B.9}$$

$P_r^{R_1}[\cdot]$ denotes the probability measured in the martingale measure R_1 which is defined by the Radon-Nikodym derivative $\dfrac{dR_1}{dQ^T} = e^{-\frac{1}{2}\int_t^T \|\gamma_{Lf}^\delta(u,T)\|^2 du + \int_t^T \gamma_{Lf}^\delta(u,T) \cdot dW_u^T}$.

From the Radon-Nikodym derivative $\dfrac{dR_1}{dQ^T}$, we know that

$$dW_t^{R_1} = dW_t^T - \gamma_{Lf}^\delta(t,T) dt. \tag{B.10}$$

Under the measure R_1, we obtain the following results substituting (B.10) into (B.6) .

$$L_f^{\delta}(T,T)=L_f^{\delta}(t,T)e^{\int_t^T\left[\overline{\mu_{2}}_f^{\delta}(u,T,T+\delta)+\frac{1}{2}\lVert\gamma_{Lf}^{\delta}(u,T)\rVert^2\right]du+\int_t^T\gamma_{Lf}^{\delta}(u,T)\cdot dW_u^{R_1}}, \tag{B.11}$$

$$L_f^{\eta}(T,T)=L_f^{\eta}(t,T)e^{\int_t^T\left[\overline{\mu_{2}}_f^{\eta}(u,T,T+\eta)-\frac{1}{2}\lVert\gamma_{Lf}^{\eta}(u,T)\rVert^2+\gamma_{Lf}^{\delta}(u,T)\cdot\gamma_{Lf}^{\eta}(u,T)\right]du+\int_t^T\gamma_{Lf}^{\eta}(u,T)\cdot dW_u^{R_1}}. \tag{B.12}$$

By inserting (B.11) and (B.12) into $P_r^{R_1}[\cdot]$, the probability can be obtained (after rearrangement) as follows:

$$P_r^{R_1}\left[L_f^{\delta}(T,T)\geq L_f^{\eta}(T,T)\middle|F_t\right]=P_r^{R_1}\left[\ln L_f^{\delta}(T,T)\geq \ln L_f^{\eta}(T,T)\middle|F_t\right]=N(d_{21}) \tag{B.13}$$

where

$$d_{21}=\frac{\ln\left(\frac{L_f^{\delta}(t,T)}{L_f^{\eta}(t,T)}\right)+\int_t^T\left[\overline{\mu_{2}}_d^{\delta}(u,T,T+\delta)-\overline{\mu_{2}}_f^{\eta}(u,T,T+\eta)\right]du+\frac{1}{2}V_2^2}{V_2}, \tag{B.14}$$

$$V_2^2=\int_t^T\lVert\gamma_{Lf}^{\delta}(u,T)-\gamma_{Lf}^{\eta}(u,T)\rVert^2\,du\,. \tag{B.15}$$

The solution of (B-II) can be derived by employing the same procedures and methods used for solving (B-I). Accordingly, the result is directly shown without expressing the details of the deriving processes.

By substituting (B.6) into (B-II), (B-II) is obtained as below:

$$(B\text{-}II)=L_f^{\eta}(t,T)e^{\int_t^T\overline{\mu_{2}}_f^{\eta}(u,T,T+\eta)du}P_r^{R_2}\left[L_f^{\delta}(T,T)\geq L_f^{\eta}(T,T)\middle|F_t\right]=L_f^{\eta}(t,T)e^{\int_t^T\overline{\mu_{2}}_f^{\eta}(u,T,T+\eta)du}N(d_{22}) \tag{B.16}$$

$$d_{22}=d_{21}-V_2 \tag{B.17}$$

$P_r^{R_2}[\cdot]$ denotes the probability measured in the martingale measure R_2 which is defined by the Radon-Nikodym derivative $\frac{dR_2}{dQ^T}=e^{-\frac{1}{2}\int_t^T\lVert\gamma_{Lf}^{\eta}(u,T)\rVert^2du+\int_t^T\gamma_{Lf}^{\eta}(u,T)\cdot dW_u^T}$.

By combining B(3), B(9) and B(13) with B(16), equation (3.2.2) of Theorem 3.2 is derived.

B.2 Proof of Equation (3.2.3)

By definition,

$$L_f^{\delta}(t,T)=\frac{1}{\delta}\left(\frac{P_f(t,T)}{P_f(t,T+\delta)}-1\right) \tag{B.18}$$

$$L_f^{\eta}(t,T)=\frac{1}{\eta}\left(\frac{P_f(t,T)}{P_f(t,T+\eta)}-1\right) \tag{B.19}$$

By substituting (B.18) and (B.19) into (3.2.2) and rearranging it, equation (3.2.3) is derived.

Appendix C: The Market Data

Exhibits 5 to 10 are drawn and computed from the DataStream database and used for the numerical example in the fourth section.

Exhibit 5: **The Exchange Rate**

Date	**2006/1/2**	**2006/7/3**	**2007/1/1**	**2007/7/2**
UK/US	1.7226	1.8407	1.95795	2.0162

The U.K./U.S. exchange rates are presented semiannually for the past 2 years.

Exhibit 6: Domestic Cap Volatilities Quoted in the U.S. Market

Date	**2006/1/2**	**2006/7/3**	**2007/1/1**	**2007/7/2**
1	11.3	10.49	11.19	8.38
2	15.62	13.07	14.75	12.43
3	17.81	14.33	15.99	13.93

The quoted volatilities of the caps in the U.K. market are presented semiannually for the past 2 years.

Exhibit 7: Foreign Cap Volatilities Quoted in the U.K. Market

Date	**2006/1/2**	**2006/7/3**	**2007/1/1**	**2007/7/2**
1	11.58	8.6	8.02	8.19
2	14.06	10.9	10.65	10.34
3	14.75	11.93	11.47	11.23

The quoted volatilities of the caps in the U.K. market are presented semiannually for the past 2 years.

Exhibit 8: Initial Domestic Forward LIBOR Rates

Date	**2006/1/2**	**2006/7/3**	**2007/1/1**	**2007/7/2**
0.0	4.839	5.808	5.564	5.587
0.5	5.014	5.908	5.412	5.485
1.0	5.008	5.783	5.205	5.456
1.5	5.058	5.717	5.001	5.388
2.0	4.928	5.786	5.128	5.571
2.5	4.896	5.762	5.045	5.603
3.0	5.018	5.860	5.196	5.693

The domestic initial forward LIBOR rates in the U.S. market are presented semiannually for the past 2 years. The rates are obtained from the associated bond prices derived from the zero curves obtained in DataStream.

Exhibit 9: Initial Foreign Forward LIBOR Rates

Date	**2006/1/2**	**2006/7/3**	**2007/1/1**	**2007/7/2**
0.0	4.699	4.943	5.577	6.316
0.5	4.562	5.235	5.693	6.516
1.0	4.630	5.424	5.675	6.533
1.5	4.699	5.477	5.708	6.505
2.0	4.713	5.432	5.599	6.463
2.5	4.713	5.496	5.573	6.467
3.0	4.679	5.356	5.448	6.329

The foreign initial forward LIBOR rates in the U.K. market are presented semiannually for the past 2 years. The rates are obtained from the associated bond prices derived from the zero curves obtained in DataStream.

Exhibit 10: The Three-Factor *B* Matrix

	Factor 1	Factor 2	Factor 3
$L_d\,(0, 0.5)$	0.8072	0.508	0.3007
$L_d\,(0,1.0)$	0.476	0.8485	- 0.2312
$L_d\,(0,1.5)$	0.2997	0.8126	- 0.4998
$L_d\,(0,2.0)$	0.0972	0.7492	- 0.6551
$L_d\,(0,2.5)$	0.2962	0.7678	- 0.5682
$L_d\,(0,3.0)$	0.2545	0.739	- 0.6239
$L_f\,(0,0.5)$	0.3355	0.7463	- 0.5749
$L_f\,(0,1.0)$	0.8557	- 0.4981	0.1406
$L_f\,(0,1.5)$	0.9273	- 0.3631	0.0907
$L_f\,(0,2.0)$	0.9608	- 0.2767	- 0.0176
$L_f\,(0,2.5)$	0.9428	- 0.2783	- 0.1836
$L_f\,(0,3.0)$	0.9566	- 0.2546	- 0.1417
X	0.9605	- 0.2002	- 0.1932

The matrix B is computed based on the correlation matrix of the relevant variables calculated from the data of the period January 2, 2006/July 2, 2007.

Notes

[1] We call, respectively, the BGM and the HJM model that are extended to a cross-currency economy and include the exchange rate dynamics in the model setting *the cross-currency BGM model and the cross-currency HJM model*.

[2] $\mu_{f_k}(t,T):\{(t,s):0\le t\le s\le T\}\times\Omega\to\Re$ is jointly measurable, adapted and $\int_0^T\left|\mu_{f_k}(u,T)\right|du<\infty \quad a.e. \quad P.$

$\sigma_{fki}:\{(t,s):0\le t\le s\le T\}\times\Omega\to\Re$ are jointly measurable, adapted and $\int_0^T\sigma_{fki}(u,T)^2\,du<\infty \quad a.e. \quad P. \quad for \quad i=1,2,...,m$.

[3] $\sigma_{Xi}:[0,\tau]\to\Re$ is deterministic for $i=1,2,...,m$. $\mu_X:[0,\tau]\to\Re$ is adapted, jointly measurable and satisfied $E\left[\int_0^T\left|\mu_X(u)\right|^2du\right]<\infty$.

[4] The Fundamental Theorem of Asset Pricing indicates that if there exists a unique martingale measure, then the economy is complete and arbitrage-free.

[5] We can do it as well from the foreign perspective.

[6] We do not report here to keep the paper within a reasonable length. The result is available upon request from the authors.

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